

# Lightface mice with finitely many Woodin cardinals from optimal determinacy hypotheses

Yizheng Zhu

Institut für mathematische Logik und Grundlagenforschung  
Fachbereich Mathematik und Informatik  
Universität Münster  
Einsteinstr. 62  
48149 Münster, Germany

October 10, 2016

## Abstract

The determinacy of lightface  $\Delta_{2n+2}^1$  and boldface  $\mathbf{\Pi}_{2n+1}^1$  sets implies the existence of an  $(\omega, \omega_1)$ -iterable  $M_{2n+1}^\#$ .

## 1 Introduction

We prove the following theorem on the equivalence of determinacy principles and the existence of an iterable mouse with an odd number of Woodin cardinals:

**Theorem 1.1.** *Suppose  $n$  is a natural number. The following are equivalent:*

1.  $\mathbf{\Pi}_{2n+1}^1$ -determinacy +  $\Delta_{2n+2}^1$ -determinacy.
2.  $\forall x \in \mathbb{R}$  (there is an  $(\omega_1, \omega_1)$ -iterable  $M_{2n}^\#(x)$ ) and there is  $N \in HC$  such that  $L[N] \models$  “there are  $2n + 1$  Woodin cardinals”.
3. There is an  $(\omega, \omega_1)$ -iterable  $M_{2n+1}^\#$ .

Theorem 1.1 solves the conjecture in [10, Section 4.2] for odd  $n$ . The new ingredient in this paper is the direction  $2 \Rightarrow 3$  for  $n > 0$ . The proof of  $3 \Rightarrow 1$  appears in [8, 9];  $1 \Rightarrow 2$  appears in [10];  $2 \Rightarrow 3$  for  $n = 0$  appears in [12].

In the proof of  $2 \Rightarrow 3$  for  $n = 0$  in [12], the key idea of producing an iterable  $M_1^\#$  is the “bad sequence argument”: If  $(\mathcal{T}_i : i < \omega)$  is a stack of iteration trees on  $\mathcal{N}$  according to an iteration strategy,  $\mathcal{N}_i$  is the last model of  $\mathcal{T}_i$  and  $\alpha \in \mathcal{N}_i$  for any  $i$ , then for all but finitely  $i$ ,  $\pi^{\mathcal{T}_i}$  exists and  $\pi^{\mathcal{T}_i}(\alpha) = \alpha$ . In practice, we take  $\alpha$  to be the Gödel code of  $(u_1, \dots, u_m)$  for any finite  $m$  in order to get proper class models whose iteration strategies respect  $(u_1, \dots, u_m)$ , and finally by varying  $m$ , the pseudo-comparison of these proper class models leads to an iterable mouse with a sharp on top of a Woodin cardinal.

This paper generalizes the “bad sequence argument” to the higher levels in the projective hierarchy. The main obstacle was the following: Say  $n = 1$ . The set of reals coding countable initial segments of  $L$  has complexity  $\Pi_1^1$ . However, the set of reals coding countable initial segments of  $M_2$  is not  $\Pi_3^1$ ! Due to this problem in complexity, the usual indiscernability arguments does not work any more with indiscernibles of  $M_2$  *above* the Woodin cardinals of  $M_2$ . Here is the correct intuition: The correct higher level analog of  $L$  is not  $M_2$ , but rather  $L[T_3]$ , where  $T_3$  is the Moschovakis tree on  $\omega \times \delta_3^1$  arising from the  $\Pi_3^1$ -scale on the universal  $\Pi_3^1$  set. “countable initial segments of  $L$ ” should correspond to  $\Pi_3^1$ -iterable mice, as defined in [11].  $\Pi_3^1$ -iterable mice is precisely the collection of mice that are strictly smaller than  $M_2|_{\delta^{M_2}}$  in the Dodd-Jensen prewellordering of mice, where  $\delta^{M_2}$  is the smallest Woodin of  $M_2$ . Instead of working with indiscernibles of  $M_2$  *above* its Woodins, one needs to work with indiscernibles for  $L[T_3]$ , or essentially, indiscernibles for iterates of  $M_2$  *below* their Woodins. The indiscernibles for  $L[T_3]$  and its relationship with  $M_2^\#$  is worked out in [16].

We briefly recall the background knowledge. Assume  $\Pi_2^1$ -determinacy. Moschovakis [7] shows that  $\Pi_3^1$  has the scale property.  $T_3$  is the tree of the  $\Pi_3^1$ -scale on the universal  $\Pi_3^1$  set. Steel [11] defines the notion of  $\Pi_3^1$ -iterable mouse. In this paper,  $\Pi_3^1$ -iterable mice are by default countable and 2-small. For any real  $x$ , the set of reals coding  $\Pi_3^1$ -iterable  $x$ -mice is  $\Pi_3^1(x)$ , uniformly in  $x$ .  $\Pi_3^1$ -iterable  $x$ -mice are genuinely  $(\omega_1, \omega_1)$ -iterable. If  $\mathcal{M}$  and  $\mathcal{N}$  are both  $\Pi_3^1$ -iterable  $x$ -mouse,  $\mathcal{M} \leq_{DJ(x)} \mathcal{N}$  means that in the comparison between  $\mathcal{M}$  and  $\mathcal{N}$ , the main branch on the  $\mathcal{M}$ -side does not drop.  $\leq_{DJ(x)}$  is a prewellordering on the set of  $\Pi_3^1$ -iterable  $x$ -mouse. We denote by

$$\|\mathcal{M}\|_{DJ(x)}$$

the  $\leq_{DJ(x)}$ -rank of  $\mathcal{M}$ . The length of  $\leq_{DJ(x)}$  is at most  $\delta_3^1$ . If  $\mathcal{M}$  is a  $\Pi_3^1$ -

iterable  $x$ -mouse, the following sets are  $\Delta_3^1(x)$ , uniformly in  $x$ :

$$\begin{aligned} &\{z : z \text{ codes a } \Pi_3^1\text{-iterable } x\text{-mouse } \mathcal{N}_z \wedge \|\mathcal{N}_z\|_{DJ(x)} < \|\mathcal{M}\|_{DJ(x)}\}, \\ &\{z : z \text{ codes a } \Pi_3^1\text{-iterable } x\text{-mouse } \mathcal{N}_z \wedge \|\mathcal{N}_z\|_{DJ(x)} = \|\mathcal{M}\|_{DJ(x)}\}. \end{aligned}$$

If  $\mathcal{M}$  is a  $\Pi_3^1$ -iterable  $x$ -mouse,

$$\mathcal{M}_\infty^x$$

denotes the direct limit of all countable non-dropping iterates of  $\mathcal{M}$  and

$$\pi_{\mathcal{M},\infty}^x : \mathcal{M} \rightarrow \mathcal{M}_\infty^x$$

denotes the direct limit map.  $\mathcal{M}_\infty^x$  depends only on  $x$  and  $\|\mathcal{M}\|_{DJ(x)}$ , so for  $\alpha = \|\mathcal{M}\|_{DJ(x)}$ , we denote

$$\mathcal{N}_{\alpha,\infty}^x = \mathcal{M}_\infty^x.$$

If  $A$  is a countable self-wellordered set, we can make sense of  $\Pi_3^1$ -iterable  $A$ -mice and  $\leq_{DJ(A)}$ ,  $\|\cdot\|_{DJ(A)}$ ,  $\mathcal{M}_\infty^A$ ,  $\pi_{\mathcal{M},\infty}^A$ ,  $\mathcal{N}_{\alpha,\infty}^A$ . As a consequence of Silver's dichotomy on  $\Delta_3^1$ -equivalence relations (cf. [3], [16, Corollary 2.14]) and  $Q$ -theory (cf. [5, 1, 4]), we are able to compare the Dodd-Jensen rank of  $\Pi_3^1$ -mice over different reals in a  $\Sigma_4^1$  way that is absolute between transitive models closed under the  $M_1^\#$ -operator:

**Theorem 1.2** ([16, Corollary 2.15]). *Assume  $\Delta_2^1$ -determinacy. Then the relations*

$$\begin{aligned} &z \text{ codes a } \Pi_3^1\text{-iterable } x\text{-mouse } \mathcal{P}_z \wedge z' \text{ codes a } \Pi_3^1\text{-iterable } x'\text{-mouse } \mathcal{P}_{z'} \\ &\wedge \|\mathcal{P}_z\|_{DJ(x)} = \|\mathcal{P}_{z'}\|_{DJ(x')} \end{aligned}$$

and

$$\begin{aligned} &z \text{ codes a } \Pi_3^1\text{-iterable } x\text{-mouse } \mathcal{P}_z \wedge z' \text{ codes a } \Pi_3^1\text{-iterable } x'\text{-mouse } \mathcal{P}_{z'} \\ &\wedge \|\mathcal{P}_z\|_{DJ(x)} = \|\mathcal{P}_{z'}\|_{DJ(x')} \wedge \\ &m \in \omega \text{ codes } \alpha \in \mathcal{P}_z \text{ relative to } z \wedge m' \in \omega \text{ codes } \alpha' \in \mathcal{P}_{z'} \text{ relative to } z' \\ &\wedge \pi_{\mathcal{P}_z,\infty}^x(\alpha) = \pi_{\mathcal{P}_{z'},\infty}^{x'}(\alpha') \end{aligned}$$

are both  $\Sigma_4^1$  and absolute between transitive models which contain  $\{z, x, z', x'\}$  and are closed under the  $M_1^\#$ -operator.

Assume  $\forall x \in \mathbb{R}$  (there is an  $(\omega_1, \omega_1)$ -iterable  $M_2^\#$ ). Steel [13] shows that:

1. For any real  $x$ ,  $\leq_{DJ(x)}$  has length  $\delta_3^1$ .

2. Let  $M_{2,\infty}^\#(x)$  be the direct limit of all countable non-dropping iterates of  $M_2^\#(x)$ . Then  $\delta_3^1$  is the least strong up to the least Woodin in  $M_{2,\infty}^\#(x)$  and  $M_{2,\infty}^\#(x) = L_{\delta_3^1}[T_3, x]$ .

We say that a  $\Pi_3^1$ -iterable  $x$ -mouse  $\mathcal{P}$  is *full* iff for any  $\Pi_3^1$ -iterable  $\mathcal{P}$ -mouse  $\mathcal{R}$ ,  $\mathcal{R}$  can be regarded as an  $x$ -mouse, i.e., for any  $\rho < o(\mathcal{P})$ , for any  $A \subseteq \rho$ ,  $A \in \mathcal{P}$  iff  $A \in \mathcal{R}$ . Equivalently,  $\mathcal{P}$  is full iff  $M_2(\mathcal{P})$  does not contain bounded subsets of  $o(\mathcal{P})$  that are not in  $\mathcal{P}$ . If  $\mathcal{P}$  is full, then  $\mathcal{P}_\infty^x = M_{2,\infty}^\#(x)|_\gamma$  where  $\gamma = o(\mathcal{P}_\infty^x)$  is a cardinal and cutpoint in  $M_{2,\infty}^\#(x)$ .

Put  $\mathbb{L}[T_3] = \bigcup_{x \in \mathbb{R}} \mathbb{L}[T_3]$ . The following theorem shows the equivalence of  $L[T_3, x]$ -indiscernibles and  $M_2^\#(x)$ :

**Theorem 1.3** (Zhu [16]). *There are countably complete  $\mathbb{L}[T_3]$ -measures  $(\mu_n : n < \omega)$  on  $(\delta_3^1)^2$  such that for any  $x \in \mathbb{R}$ ,*

1. *for  $\mu_n$ -a.e.  $(\alpha, \beta)$ , if  $\|\mathcal{R}\|_{DJ(x)} = \beta$ , then  $\mathcal{R}$  is full and  $\mathcal{R}_\infty = \mathcal{N}_{\beta,\infty}^x = M_{2,\infty}^\#(x)|_\beta$ ;*

2. *letting*

$$(x^{3\#})_n = \{\ulcorner \varphi^\top : \text{for } \mu_n\text{-a.e. } (\alpha, \beta), \mathcal{N}_{\beta,\infty}^x \models \varphi(\alpha) \}$$

*and*

$$x^{3\#} = \oplus_{n < \omega} (x^{3\#})_n,$$

*then*

$$x^{3\#} \equiv_m M_2^\#(x),$$

*uniformly in  $x$ .*

Fixing  $n$ ,  $\mu_n$  is the higher level analog of the  $\mathbb{L} =_{\text{DEF}} \bigcup_{x \in \mathbb{R}} L[x]$ -measure  $\nu_n$  on  $(\omega_1)^n$ , where  $A \in \nu_n$  iff for some  $x \in \mathbb{R}$ ,  $A$  contains all the increasing  $n$ -tuples of countable  $x$ -indiscernibles. For the reader familiar with [16],  $\mu_n$  can be taken as the  $\mathbb{L}[T_3]$ -measure arising from the level-3 tree  $Y_n$  so that  $\llbracket \emptyset \rrbracket_{Y_n} = u_n + \omega$ .

## 2 The bad sequence argument

We prove  $2 \Rightarrow 3$  in Theorem 1.1 for  $n = 1$ . The general case makes no essential difference based on [16].

**Definition 2.1.** A premouse  $\mathcal{P}$  is *suitable* iff there is  $\delta \in \mathcal{P}$  such that

1.  $\mathcal{P} = M_2^\#(\mathcal{P}|\delta)|(\delta^+)^{M_2^\#(\mathcal{P}|\delta)}$ .

2.  $\mathcal{P}$  satisfies the following.

- (a)  $\delta$  is Woodin.
- (b)  $\forall \eta < \delta \forall a \in \mathcal{P}|\eta$  (the  $L[\vec{E}]$ -construction above  $a$  with critical points above  $\eta$  reaches  $M_2^\#(a)$ ). If  $\mathcal{N}$  is a  $\text{card}(\mathcal{N})^+ + 1$ -iterable mouse (over  $\emptyset$ ), then let

$$M_2^*(\mathcal{N}) = M_2^\#(\mathcal{N})|_\alpha$$

regarded as a  $\emptyset$ -mouse, where  $\alpha$  is the least such that  $\exists \rho < o(\mathcal{N}) \exists A \subseteq \rho (A \in \text{rud}(M_2^\#(\mathcal{N})|_\alpha) \setminus \mathcal{N})$ . The *partial iteration strategy guided by 2-small mice* is the partial strategy  $\Sigma$  so that  $\Sigma(\mathcal{T}) = b$  iff  $\mathcal{Q}(b, \mathcal{T}) = M_2^*(\mathcal{M}(\mathcal{T})) \neq M_2^\#(\mathcal{M}(\mathcal{T}))$ .

- (c)  $\forall \eta < \delta \mathcal{P}|\delta$  is  $(\eta, \eta)$ -iterable according to the partial iteration strategy guided by 2-small mice.
- (d)  $\forall \eta < \delta M_2^\#(\mathcal{P}|\eta) \models \eta$  is not Woodin.

If  $\mathcal{P}$  is suitable,  $\delta^\mathcal{P}$  denotes its Woodin, and  $\mathcal{P}^- = \mathcal{P}|\delta^\mathcal{P}$ . If  $\mathcal{P}$  is also countable,  $\mathcal{P}$  itself can be regarded as a full  $\Pi_3^1$ -iterable  $\mathcal{P}^-$ -mouse. In fact, a countable premouse  $\mathcal{P}$  is suitable iff  $\mathcal{P}$  satisfies the first order property in Clause 2 in Definition 2.1 and  $\mathcal{P}$  is full. If  $\hat{\mathcal{P}}$  is another  $\Pi_3^1$ -iterable  $\mathcal{P}^-$ -mouse,  $\hat{\mathcal{P}}$  can also be regarded as a  $\emptyset$ -premouse, and we have  $\mathcal{P} \leq \hat{\mathcal{P}}$  iff  $\hat{\mathcal{P}}$  is full.

**Theorem 2.2** (Mitchell-Steel [6]). *If  $\forall x \in \mathbb{R}$  (there is an  $(\omega_1, \omega_1)$ -iterable  $M_2^\#(x)$ ) and  $\exists N \in HC \ L[N] \models$  “there are three Woodin cardinals”, then there is a countable suitable premouse.*

The following condensation principle is an easy generalization of [12, Lemma 3.3]. Its proof can be found in e.g. [10, Lemma 3.3].

**Lemma 2.3.** *If  $\mathcal{P}$  is countable and suitable,  $\hat{\mathcal{P}}$  is a  $\Pi_3^1$ -iterable  $\mathcal{P}^-$ -mouse,  $\mathcal{H}$  is the transitive collapse of  $\text{Hull}_\omega^{\hat{\mathcal{P}}}$ , the  $\hat{\mathcal{P}}$ -definable points where  $\hat{\mathcal{P}}$  is regarded as a  $\emptyset$ -premouse, then  $\mathcal{H}$  (regarded as a  $\emptyset$ -premouse) is an initial segment of  $\mathcal{P}|\omega_1^\mathcal{P}$ .*

**Definition 2.4.** Let  $\mathcal{T}$  be a normal iteration tree on a suitable  $\mathcal{P}$ .  $\mathcal{T}$  is *short* iff  $\forall \alpha \leq \text{lh}(\mathcal{T})$  limit,  $M_2(\mathcal{M}(\mathcal{T} \upharpoonright \alpha)) \models$  “ $\delta(\mathcal{T} \upharpoonright \alpha)$  is not Woodin”.  $\mathcal{T}$  is *maximal* iff  $\mathcal{T}$  is not short.

**Definition 2.5.** Suppose  $\mathcal{P}$  is suitable.  $\mathcal{P}$  is *short-tree-iterable* iff for any putative short tree  $\mathcal{T}$  on  $\mathcal{P}$ , for any  $\Pi_3^1$ -iterable  $\mathcal{P}^-$ -mouse  $\hat{\mathcal{P}}$ , letting  $\hat{\mathcal{T}}$  be  $\mathcal{T}$  construed as a putative tree on  $\hat{\mathcal{P}}$ ,

1. if  $\widehat{\mathcal{T}}$  has a last model  $\mathcal{M}_\alpha^{\widehat{\mathcal{T}}}$ , then either
  - (a)  $[0, \alpha]_T$  drops,  $\mathcal{M}_\alpha^{\mathcal{T}}$  is  $\Pi_3^1$ -iterable, or
  - (b)  $[0, \alpha]_T$  does not drop,  $\mathcal{M}_\alpha^{\widehat{\mathcal{T}}}$  is a  $\Pi_3^1$ -iterable  $\pi_{0,\alpha}^{\mathcal{T}}(\mathcal{P}^-)$ -mouse.
2. If  $\text{lh}(\mathcal{T})$  is limit,  $\mathcal{T}$  is short, then  $\mathcal{T}$  has a cofinal branch  $b$  such that  $\mathcal{Q}(b, \mathcal{T}) = M_2^*(\mathcal{M}(\mathcal{T}))$ , where  $M_2^*(\mathcal{M}(\mathcal{T})) = M_2^\#(\mathcal{M}(\mathcal{T}))|_\alpha$  regarded as a  $\emptyset$ -mouse,  $\alpha$  is the least such that  $\exists \rho < \delta(\mathcal{T}) \exists A \subseteq \rho$  ( $A \in M_2^\#(\mathcal{M}(\mathcal{T}))|_{\alpha+1} \setminus \mathcal{M}(\mathcal{T})$ ).

**Lemma 2.6.** *If  $\mathcal{P}$  is suitable, then  $\mathcal{P}$  is short-tree-iterable.*

*Proof.* Suppose not. There is then a putative short tree  $\mathcal{T}$  on  $\mathcal{P}$  and a  $\Pi_3^1$ -iterable  $\mathcal{P}^-$ -mouse  $\widehat{\mathcal{P}}$  such that either

1.  $\text{lh}(\mathcal{T}) = \alpha + 1$  is a successor,  $[0, \alpha]_T$  drops,  $\mathcal{M}_\alpha^{\mathcal{T}}$  is not  $\Pi_3^1$ -iterable, or
2.  $\text{lh}(\mathcal{T}) = \alpha + 1$  is a successor,  $[0, \alpha]_T$  does not drop, letting  $\widehat{\mathcal{T}}$  be  $\mathcal{T}$  construed as a putative tree on  $\mathcal{P}$ , then  $\mathcal{M}_\alpha^{\widehat{\mathcal{T}}}$  is not a  $\Pi_3^1$ -iterable  $\pi_{0,\alpha}^{\mathcal{T}}(\mathcal{P}^-)$ -mouse, or
3.  $\text{lh}(\mathcal{T}) = \lambda$  is a limit, there is a  $\delta(\mathcal{T})$ -sound,  $\Pi_3^1$ -iterable  $\mathcal{M}(\mathcal{T})$ -mouse  $\mathcal{R}$  that can be regarded as an  $\emptyset$ -premouse with  $\rho_\omega(\mathcal{R}) < \delta(\mathcal{T})$ , but there is no cofinal branch  $b$  such that  $\mathcal{Q}(b, \mathcal{T}) = \mathcal{R}$ .

The existence of a  $\mathcal{P}$ -bad pair  $(\mathcal{T}, \widehat{\mathcal{P}})$  is  $\Sigma_4^1$  in the code of  $\mathcal{P}$ . By Steel [11],  $M_2(z) \prec_{\Sigma_4^1} V$  for any real  $z$ . Hence, a bad pair can be found in  $M_2(\mathcal{P})^{\text{Coll}(\omega, \mathcal{P})}$ . Working in  $M_2(\mathcal{P})$ , take a countable elementary substructure  $\mathcal{N} \prec M_2(\mathcal{P})|_\eta$ , where  $\eta$  is the successor of  $o(\mathcal{P})$  in  $M_2(\mathcal{P})$ .  $\mathcal{H}$  is the transitive collapse of  $\mathcal{N}$ , which is by Lemma 2.3 an initial segment of  $\mathcal{P}$ . Let  $\mathcal{Q}$  be the image of  $\mathcal{P}$  under the transitive collapsing map. Take  $g \in \mathcal{P}$  which is generic over  $\mathcal{H}$  for  $\text{Coll}(\omega, \mathcal{Q})$ . So  $\mathcal{H}[g] \models$  “there is a  $\mathcal{Q}$ -bad pair  $(\mathcal{U}, \widehat{\mathcal{Q}})$ ”. Note that  $\mathcal{H}[g] \models$  “I am closed under the  $M_1^\#$ -operator”, therefore as  $\mathcal{H} \leq \mathcal{P}$ , the  $M_1^\#$ -operators are computed correctly in  $\mathcal{H}[g]$ , which implies that  $\mathcal{H}[g] \prec_{\Sigma_3^1} \mathcal{P}$  by genericity iterations (cf. [14, Section 7.2]). So  $(\mathcal{U}, \widehat{\mathcal{Q}})$ , being a  $\mathcal{Q}$ -bad pair from the point of view of  $\mathcal{H}[g]$ , is also seen as a  $\mathcal{Q}$ -bad pair in  $\mathcal{P}$ . However,  $\mathcal{Q} \triangleleft \mathcal{P}$  and  $\mathcal{Q}$  is  $(\omega_1, \omega_1)$ -iterable in  $\mathcal{P}$  by suitability. Contradiction!  $\square$

If  $\mathcal{P}$  is suitable and  $\mathcal{T}$  is a short tree on  $\mathcal{P}$  such that  $\pi^{\mathcal{T}}$  exists, we can define an order preserving function

$$g^{\mathcal{T}} : \delta_3^1 \rightarrow \delta_3^1$$

as follows: If  $\mathcal{P}'$  is a  $\Pi_3^1$ -iterable  $\mathcal{P}^-$ -mouse, let  $f^\mathcal{T}(\mathcal{P}')$  be the last model of  $\mathcal{T}$  construed as a tree on  $\mathcal{P}'$ , and define

$$g^\mathcal{T}(\|\mathcal{P}'\|_{\mathcal{P}^-}) = \|f^\mathcal{T}(\mathcal{P}')\|_{\pi^\mathcal{T}(\mathcal{P}^-)}.$$

$g^\mathcal{T}$  is well-defined: Suppose  $\|\mathcal{P}'\|_{\mathcal{P}^-} = \|\mathcal{P}''\|_{\mathcal{P}^-}$  and suppose without loss of generality that  $\mathcal{P}''$  is a nondropping iterate of  $\mathcal{P}'$  via  $\mathcal{U}$  above  $\mathcal{P}^-$ . We would like to show that  $\|f^\mathcal{T}(\mathcal{P}')\|_{\pi^\mathcal{T}(\mathcal{P}^-)} = \|f^\mathcal{T}(\mathcal{P}'')\|_{\pi^\mathcal{T}(\mathcal{P}^-)}$ . On the one hand, the tree  $\mathcal{P}'$ -to- $f(\mathcal{P}')$  is copied to the tree  $\mathcal{P}''$ -to- $f(\mathcal{P}'')$  according to  $\pi^\mathcal{U}$  (both trees are just  $\mathcal{T}$  construed on different models), so  $\pi^\mathcal{U}$  induces a copying map from  $f^\mathcal{T}(\mathcal{P}')$  to  $f^\mathcal{T}(\mathcal{P}'')$ , giving that  $\|f^\mathcal{T}(\mathcal{P}')\|_{\pi^\mathcal{T}(\mathcal{P}^-)} \leq \|f^\mathcal{T}(\mathcal{P}'')\|_{\pi^\mathcal{T}(\mathcal{P}^-)}$ . On the other hand, we can copy  $\mathcal{U}$  to a tree on  $f(\mathcal{P}')$  according to the iteration map from  $\mathcal{P}'$  to  $f(\mathcal{P}')$ , leading to an iteration tree  $\mathcal{V}$  on  $f(\mathcal{P}')$  with last model  $\mathcal{Q}$  so that  $\pi^\mathcal{V}$  exists. Note that  $\mathcal{U}$  is above  $\mathcal{P}^-$  while  $\mathcal{T}$  is based on  $\mathcal{P}^-$ . The technique in [15, Lemma 3.2] enables us to define a map from  $f(\mathcal{P}'')$  to  $\mathcal{Q}$ , giving that  $\|f^\mathcal{T}(\mathcal{P}'')\|_{\pi^\mathcal{T}(\mathcal{P}^-)} \leq \|f^\mathcal{T}(\mathcal{P}')\|_{\pi^\mathcal{T}(\mathcal{P}^-)}$ . A similar argument shows that  $g^\mathcal{T}$  is order preserving.

**Corollary 2.7.** *Suppose  $\mathcal{P}$  is suitable and  $\mathcal{T}$  is a short tree on  $\mathcal{P}$  with last model  $\mathcal{Q}$  such that  $\pi^\mathcal{T}$  exists. Then  $\mathcal{Q}$  is suitable.*

*Proof.* All the first-order-in- $\mathcal{P}$  properties in Definition 2.1 are preserved by elementarity. We need to show fullness. For any  $\mathcal{P}'$ , a full  $\Pi_3^1$ -iterable  $\mathcal{P}^-$ -mouse,  $\mathcal{P}' \models o(\mathcal{P}) = (\delta^\mathcal{P})^+$ , and hence by elementarity,  $f^\mathcal{T}(\mathcal{P}') \models o(\mathcal{Q}) = (\pi^\mathcal{T}(\delta^\mathcal{P}))^+$ . If  $\mathcal{Q}'$  is any full  $\Pi_3^1$ -iterable  $\mathcal{Q}|\pi^\mathcal{T}(\delta^\mathcal{P})$ -mouse, we may pick such  $\mathcal{P}'$  with  $g^\mathcal{T}(\|\mathcal{P}'\|_{\mathcal{P}^-}) > \|\mathcal{Q}'\|_{\pi^\mathcal{T}(\mathcal{P}^-)}$ , implying that  $\mathcal{Q}' \models o(\mathcal{Q}) = (\pi^\mathcal{T}(\delta^\mathcal{P}))^+$ . Hence  $\mathcal{Q}$  is suitable and  $\delta^\mathcal{Q} = \pi^\mathcal{T}(\delta^\mathcal{P})$ .  $\square$

**Definition 2.8.** Let  $\mathcal{P}$  be suitable.  $\mathcal{Q}$  is called a *pseudo-normal-iterate* of  $\mathcal{P}$  iff  $\mathcal{Q}$  is suitable and there is a normal tree  $\mathcal{T}$  on  $\mathcal{P}$  such that either  $\mathcal{Q}$  is the last model of  $\mathcal{T}$ ,  $\pi^\mathcal{T}$  exists, or  $\mathcal{Q} = M_2(\mathcal{M}(\mathcal{T}))|((\delta(\mathcal{T}))^+)^{M_2(\mathcal{M}(\mathcal{T}))}$ .

**Definition 2.9.** Let  $\mathcal{P}$  be suitable.  $((\mathcal{T}_i : i < k), (\mathcal{P}_i : i \leq k))$  is called a *finite full stack* on  $\mathcal{P}$  iff  $\mathcal{P}_0 = \mathcal{P}$  and for each  $i$ ,  $\mathcal{P}_{i+1}$  is a pseudo-normal-iterate of  $\mathcal{P}_i$  witnessed by  $\mathcal{T}_i$ .

**Definition 2.10.** Suppose  $\mathcal{P}$  is countable and suitable,  $\alpha < \beta < \delta_3^1$ , and  $\alpha < \mathcal{N}_{\beta, \infty}^{\mathcal{P}^-}$ . Then

1.  $Th_{(\alpha, \beta)}^\mathcal{P} = \{(\ulcorner \varphi \urcorner, \xi) : \xi < \delta^\mathcal{P}, \mathcal{N}_{\beta, \infty}^{\mathcal{P}^-} \models \varphi(\xi, \alpha)\}$ .
2.  $\gamma_{(\alpha, \beta)}^\mathcal{P} = \sup(Hull^{\mathcal{N}_{\beta, \infty}^{\mathcal{P}^-}}(\{\alpha\}) \cap \delta^\mathcal{P})$ .
3.  $H_{(\alpha, \beta)}^\mathcal{P} = Hull^{\mathcal{N}_{\beta, \infty}^{\mathcal{P}^-}}(\gamma_{(\alpha, \beta)}^\mathcal{P} \cup \{\alpha\})$ .

4. By Theorem 1.3, define  $(\mathcal{P}^-)^{3\#}_n = Th_{(\alpha,\beta)}^{\mathcal{P}}$  for  $\mu_n$ -a.e.  $(\alpha, \beta)$  and  $(\mathcal{P}^-)^{3\#} = \bigoplus_{n < \omega} (\mathcal{P}^-)^{3\#}_n$ .
5.  $\gamma_{3\#,n}^{\mathcal{P}} = \gamma_{(\alpha,\beta)}^{\mathcal{P}}$  for  $\mu_n$ -a.e.  $(\alpha, \beta)$ .
6.  $H_{3\#,n}^{\mathcal{P}} = H_{(\alpha,\beta)}^{\mathcal{P}}$  for  $\mu_n$ -a.e.  $(\alpha, \beta)$ .

Suppose  $\mathcal{T}$  is a normal iteration tree on  $\mathcal{P}$  and  $b$  is a cofinal branch of  $\mathcal{T}$ .  $b$  is said to *respect*  $(\alpha, \beta)$  iff  $\mathcal{Q} = \mathcal{M}_b^{\mathcal{T}}$  is suitable and

$$\pi_b^{\mathcal{T}}(Th_{(\alpha,\beta)}^{\mathcal{P}}) = Th_{(\alpha,\beta)}^{\mathcal{Q}}.$$

$b$  is said to *respect*  $(\cdot)_n^{3\#}$  iff  $\mathcal{Q} = \mathcal{M}_b^{\mathcal{T}}$  is suitable and

$$\pi_b^{\mathcal{T}}((\mathcal{P}^-)^{3\#}_n) = (\mathcal{Q}^-)^{3\#}_n.$$

$\mathcal{P}$  is *n-iterable* iff for any finite full stack  $((\mathcal{T}_i : i < k), (\mathcal{P}_i : i \leq k))$  on  $\mathcal{P}$ , there is  $(b_i : i < k)$  such that each  $b_i$  respects  $(\cdot)_n^{3\#}$ .

By Theorem 1.3, for any countable suitable  $\mathcal{P}$ , we must have

$$\sup_{n < \omega} \gamma_{3\#,n}^{\mathcal{P}} = \delta^{\mathcal{P}}.$$

**Lemma 2.11.** *Let  $n < \omega$ . Then there is a countable, n-iterable suitable mouse.*

*Proof.* Otherwise, there is  $((\mathcal{T}_i : i < \omega), (\mathcal{P}_i : i < \omega))$  such that  $\mathcal{P}_0 = \mathcal{P}$  is suitable,  $\mathcal{T}_i$  is a normal tree on  $\mathcal{P}_i$ , but for infinitely many  $i$ , there is no cofinal branch  $b_i$  through  $\mathcal{T}_i$  that respects  $(\cdot)_n^{3\#}$ .

Fix  $z \in \mathbb{R}$  coding  $(\vec{\mathcal{T}}, \vec{\mathcal{P}})$ . Fix  $\alpha < \beta < \delta_3^1$  such that for any  $i$ ,  $(\mathcal{P}_i)^{3\#}_n = Th_{(\alpha,\beta)}^{\mathcal{P}}$  and if  $\|\mathcal{R}\|_{DJ(\mathcal{P}_i^-)} = \beta$ , then  $\mathcal{R}$  is full and  $\mathcal{R}_\infty = \mathcal{N}_{\beta,\infty}^{\mathcal{P}_i^-} = M_{2,\infty}^\#(\mathcal{P}_i^-)|\beta$ . Thus, for infinitely many  $i$ , there is no cofinal branch  $b_i$  through  $\mathcal{T}_i$  that respects  $(\alpha, \beta)$ . We call  $(\vec{\mathcal{T}}, \vec{\mathcal{P}})$  an  $(\alpha, \beta)$ -bad sequence based on  $\mathcal{P}$ .

Let  $\hat{\mathcal{P}}$  be a  $\Pi_3^1$ -iterable  $\mathcal{P}^-$ -mouse and  $\eta \in \hat{\mathcal{P}}$  so that  $\|\hat{\mathcal{P}}\|_{\mathcal{P}^-} = \beta$  and  $\pi_{\hat{\mathcal{P}},\infty}^{\mathcal{P}^-}(\eta) = \alpha$ . By Theorem 1.2, the statement

“There is an  $(\pi_{\hat{\mathcal{P}},\infty}^{\mathcal{P}^-}(\eta), \|\hat{\mathcal{P}}\|_{\mathcal{P}^-})$ -bad sequence  $(\vec{\mathcal{T}}, \vec{\mathcal{P}})$  based on  $\mathcal{P}$ ”

is  $\Sigma_4^1$  in the code of  $\hat{\mathcal{P}}$  and absolute between transitive models closed under the  $M_1^\#$ -operator. It is a true statement in  $V$ , so by absoluteness, true  $M_2(\hat{\mathcal{P}})^{Coll(\omega, \hat{\mathcal{P}})}$  as well. By our choice of  $\beta$ ,  $\hat{\mathcal{P}}$  is full, so  $M_2(\hat{\mathcal{P}})$  can be regarded as an  $\emptyset$ -premouse and  $o(\hat{\mathcal{P}})$  is a cardinal and cutpoint in  $M_2(\hat{\mathcal{P}})$ . As in the proof of Lemma 2.6, we get  $\mathcal{H} \triangleleft \mathcal{P}$  and  $g \in \mathcal{P}$  generic over  $\mathcal{H}$ ,  $\{\mathcal{Q}, \vec{\mathcal{Q}}, \vec{\mathcal{U}}, \xi\} \in \mathcal{H}[g]$  so that



$\mathcal{H}[g] \models “(\vec{\mathcal{U}}, \vec{\mathcal{Q}}) \text{ is an } (\pi_{\widehat{\mathcal{Q}}, \infty}^{\mathcal{Q}^-}(\eta), \|\widehat{\mathcal{Q}}\|_{DJ(\mathcal{Q}^-)})\text{-bad sequence based on } \mathcal{Q}.”$

As  $\mathcal{H}[g] \prec_{\Sigma_3^1} \mathcal{P}$ , we have  $(\bar{\alpha}, \bar{\beta}) \in \mathcal{P}$  so that

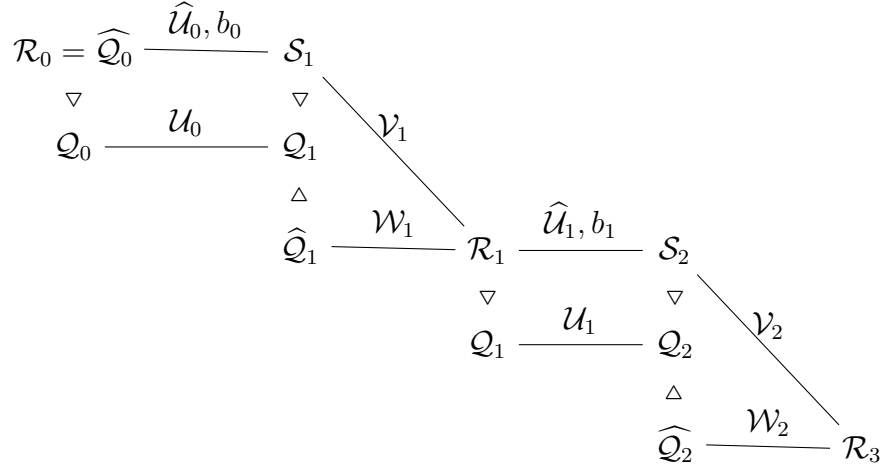
$\mathcal{P} \models “(\vec{\mathcal{U}}, \vec{\mathcal{Q}}) \text{ is an } (\bar{\alpha}, \bar{\beta})\text{-bad sequence and } \|\widehat{\mathcal{Q}}\|_{DJ(\mathcal{Q}^-)} = \bar{\beta}, \pi_{\widehat{\mathcal{Q}}, \infty}^{\mathcal{Q}^-}(\eta) = \bar{\alpha}.”$

For the rest of this proof, we work in  $\mathcal{P}$ . Pick  $\widehat{\mathcal{Q}}_i$  and  $\xi_i \in \widehat{\mathcal{Q}}_i$  so that

$$\mathcal{P} \models \|\widehat{\mathcal{Q}}_i\|_{\mathcal{Q}_i^-} = \bar{\beta} \wedge \pi_{\widehat{\mathcal{Q}}_i, \infty}^{\mathcal{Q}_i^-}(\xi_i) = \bar{\alpha}.$$

We define  $(\mathcal{R}_i, \mathcal{S}_i, b_i, \widehat{\mathcal{U}}_i : i < \omega)$  and  $(\mathcal{V}_i, \mathcal{W}_i : 1 \leq i < \omega)$  inductively such that:

1.  $\mathcal{R}_i \triangleright \mathcal{Q}_i$ ,  $\mathcal{R}_i$  is  $\Pi_3^1$ -iterable above  $\mathcal{Q}_i^-$ ,  $\mathcal{R}_0 = \widehat{\mathcal{Q}}_0$ ;
2.  $\widehat{\mathcal{U}}_i$  is  $\mathcal{U}_i$  construed as an iteration tree on  $\mathcal{R}_i$ ;
3.  $b_i$  is the cofinal branch of  $\widehat{\mathcal{U}}_i$  chosen by the internal strategy of  $\mathcal{P}$ ;
4.  $\mathcal{S}_{i+1}$  is the last model of  $\widehat{\mathcal{U}}_i \frown b_i$ ;
5.  $(\mathcal{V}_i, \mathcal{W}_i)$  is the comparison of  $(\mathcal{S}_i, \widehat{\mathcal{Q}}_i)$  and  $\mathcal{R}_i$  is the last model of  $\mathcal{W}_i$ .



By monotonicity of the function  $g^{\widehat{\mathcal{U}} \frown b_i} : \delta_3^1 \rightarrow \delta_3^1$ , we can inductively see that for each  $i$ ,  $\|\mathcal{S}_i\|_{DJ(\mathcal{Q}_i^-)} \geq \bar{\beta}$ , the main branch of  $\mathcal{W}_i$  does not drop, and  $\|\mathcal{R}_i\|_{DJ(\mathcal{Q}_i^-)} \geq \bar{\beta}$ . The stack

$$\widehat{\mathcal{U}}_0 \frown b_0 \frown \mathcal{V}_1 \frown \widehat{\mathcal{U}}_1 \frown b_1 \frown \mathcal{V}_2 \frown \dots$$

is according to the internal strategy of  $\mathcal{P}$ . So for some  $m < \omega$ , we have for any  $i > m$ ,  $\pi_{b_i}^{\mathcal{U}_i}$  exists and  $\pi^{\mathcal{V}_i}$  exists. The map  $\pi^{\hat{\mathcal{U}}_i \frown b_i \frown \mathcal{V}_{i+1}}$  induces a map  $\tau_i : (\mathcal{R}_i)_{\infty}^{\mathcal{Q}_i^-} \rightarrow (\mathcal{R}_{i+1})_{\infty}^{\mathcal{Q}_{i+1}^-}$  so that  $\tau_i \circ \pi_{\mathcal{R}_i, \infty}^{\mathcal{Q}_i^-} = \pi_{\mathcal{R}_{i+1}, \infty}^{\mathcal{Q}_{i+1}^-} \circ \pi^{\hat{\mathcal{U}}_i \frown b_i \frown \mathcal{V}_{i+1}}$ . Clearly  $\tau_i(\bar{\beta}) \geq \bar{\beta}$ . So  $\pi^{\hat{\mathcal{U}}_i \frown b_i \frown \mathcal{V}_{i+1}} \circ \pi^{\mathcal{U}_i}(\xi_i) \geq \pi^{\mathcal{U}_{i+1}}(\xi_{i+1})$ . So we must have some  $m < m' < \omega$  so that for any  $i > m'$ ,  $\pi^{\hat{\mathcal{U}}_i \frown b_i \frown \mathcal{V}_{i+1}} \circ \pi^{\mathcal{U}_i}(\xi_i) = \pi^{\mathcal{U}_{i+1}}(\xi_{i+1})$ . In other words, for any  $i > m'$ ,  $b_i$  respects  $(\bar{\alpha}, \bar{\beta})$ , contradicting to the assumption that  $(\vec{\mathcal{U}}, \vec{\mathcal{Q}})$  is an  $(\bar{\alpha}, \bar{\beta})$ -bad sequence.  $\square$

By Lemma 2.11, we can find  $(\mathcal{P}_n : n < \omega)$  where  $\mathcal{P}_n$  is a countable,  $n$ -iterable suitable premouse. The pseudo-comparison leads to countable iteration trees  $(\mathcal{T}_n : n < \omega)$  and a suitable  $\mathcal{Q}$  so that  $\mathcal{T}_n$  is an iteration tree on  $\mathcal{P}_n$  with last model  $\mathcal{Q}$ .  $\mathcal{Q}$  is then  $n$ -iterable for any  $n < \omega$ . The usual limit branching argument (cf. [12, Lemma 4.12]) gives an  $(\omega, \omega_1)$ -iteration strategy for  $\mathcal{Q}$ : For instance, suppose  $\mathcal{T}$  is a normal tree on  $\mathcal{Q}^-$  with pseudo-normal-iterate  $\mathcal{R}$ . Let

$$b_i = \cap \{b : b \text{ is a branch through } \mathcal{T} \wedge b \text{ respects } (\cdot)_i^{3\#}\}.$$

Then  $b_i \subseteq b_{i+1}$ ,  $\gamma_{3\#, i}^{\mathcal{M}_{\max b_i}^{\mathcal{T}}} = \gamma_{3\#, i}^{\mathcal{R}}$ , and we have an isomorphism  $\sigma_i : H_{3\#, i}^{\mathcal{M}_{\max b_i}^{\mathcal{T}}} \cong H_{3\#, i}^{\mathcal{R}}$ . Let  $b = \cup_{i < \omega} b_i$ . Then  $\delta^{\mathcal{M}_{\sup b}^{\mathcal{T}}} \geq \sup_{n < \omega} \gamma_{3\#, n}^{\mathcal{R}} = \delta^{\mathcal{R}}$ . So  $b$  must be a cofinal branch. There is a canonical map

$$\tau : \mathcal{R} \rightarrow \mathcal{M}_b^{\mathcal{T}}$$

defined by  $\tau(a) = \pi_{\alpha, b} \circ \sigma_i^{-1}(a)$  for  $\alpha < \max(b_i)$  and  $a \in H_{3\#, i}^{\mathcal{R}}$ .  $\tau$  is onto  $\mathcal{M}_b^{\mathcal{T}}$  because  $\mathcal{M}_{\max b_i}^{\mathcal{T}} = \cup_{n < \omega} H_{3\#, n}^{\mathcal{M}_{\max b_i}^{\mathcal{T}}}$ . Therefore,  $\tau$  is the identity,  $\mathcal{M}_b^{\mathcal{T}} = \mathcal{R}$  and  $b$  respects  $(\cdot)_n^{3\#}$ .

In other words, by Theorem 1.3,  $M_2^{\#}(\mathcal{Q})$ , regarded as a  $\emptyset$ -mouse, has a partial  $(\omega, \omega_1)$ -iteration strategy  $\Gamma$  with respect to stacks of normal trees based on  $\mathcal{Q}$  that moves the top  $M_2^{\#}$ -component correctly, i.e., whenever  $\mathcal{U}$  is according to  $\Gamma$  based on  $\mathcal{Q}$  and the main branch of  $\mathcal{U}$  does not drop, the last model of  $\mathcal{U}$  must be  $M_2^{\#}(\pi^{\mathcal{U}}(\mathcal{Q}))$ . Also by definition of suitability, whenever  $\mathcal{U}$  is according to  $\Gamma$  based on  $\mathcal{Q}^-$  but the main branch of  $\mathcal{U}$  drops, the last model of  $\mathcal{U}$  is  $\Pi_3^1$ -iterable. By the technique in [2],  $M_2^{\#}(\mathcal{Q})$  is  $(\omega, \omega_1)$ -iterable.  $M_2^{\#}(\mathcal{Q})$  has a sharp above three Woodins, so by taking its  $\Sigma_1$ -Skolem hull, we get an  $(\omega, \omega_1)$ -iterable  $M_3^{\#}$ . This finishes the proof of Theorem 1.1 for  $n = 1$ .

## References

- [1] Howard S. Becker and Alexander S. Kechris. Sets of ordinals constructible from trees and the third Victoria Delfino problem. In *Axiomatic set theory (Boulder, Colo., 1983)*, volume 31 of *Contemp. Math.*, pages 13–29. Amer. Math. Soc., Providence, RI, 1984.
- [2] Gunter Fuchs, Itay Neeman, and Ralf Schindler. A criterion for coarse iterability. *Arch. Math. Logic*, 49(4):447–467, 2010.
- [3] Greg Hjorth. Some applications of coarse inner model theory. *J. Symbolic Logic*, 62(2):337–365, 1997.
- [4] A. S. Kechris and D. A. Martin. On the theory of  $\Pi^1_3$  sets of reals, II. In *Ordinal Definability and Recursion Theory. The Cabal Seminar. Volume III*, volume 43 of *Lect. Notes Log.*, pages 200–219. Cambridge University Press, Cambridge, 2016.
- [5] Alexander S. Kechris, Donald A. Martin, and Robert M. Solovay. Introduction to  $Q$ -theory. In *Ordinal Definability and Recursion Theory. The Cabal Seminar. Volume III*, volume 43 of *Lect. Notes Log.*, pages 126–199. Cambridge University Press, Cambridge, 2016.
- [6] William J. Mitchell and John R. Steel. *Fine structure and iteration trees*, volume 3 of *Lecture Notes in Logic*. Springer-Verlag, Berlin, 1994.
- [7] Yiannis N. Moschovakis. *Descriptive set theory*, volume 155 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2009.
- [8] Itay Neeman. Optimal proofs of determinacy. *Bull. Symbolic Logic*, 1(3):327–339, 1995.
- [9] Itay Neeman. Optimal proofs of determinacy. II. *J. Math. Log.*, 2(2):227–258, 2002.
- [10] Ralf Schindler, Sandra Uhlenbrock, and Hugh Woodin. Mice with finitely many Woodin cardinals from optimal determinacy hypotheses, available at <http://wwwmath.uni-muenster.de/u/rds/>.
- [11] J. R. Steel. Projectively well-ordered inner models. *Ann. Pure Appl. Logic*, 74(1):77–104, 1995.

- [12] J. R. Steel and W. Hugh Woodin. HOD as a core model. In *Ordinal Definability and Recursion Theory. The Cabal Seminar. Volume III*, volume 43 of *Lect. Notes Log.*, pages 257–346. Cambridge University Press, Cambridge, 2016.
- [13] John R. Steel.  $\text{HOD}^{L(\mathbb{R})}$  is a core model below  $\Theta$ . *Bull. Symbolic Logic*, 1(1):75–84, 1995.
- [14] John R. Steel. An outline of inner model theory. In *Handbook of set theory. Vols. 1, 2, 3*, pages 1595–1684. Springer, Dordrecht, 2010.
- [15] Yizheng Zhu. Realizing an  $\text{AD}^+$  model as a derived model of a premouse. *Ann. Pure Appl. Logic*, 166(12):1275–1364, 2015.
- [16] Yizheng Zhu. The higher sharp, 2016, arXiv:1604.00481.